SPRING 2024: BONUS PROBLEM 4 SOLUTION

Bonus Problem 4. Let V denote the vector space of 2×2 matrices over $\mathbb{R}$ and define $T : V \to V$ by $T(A) = A^t$. Show that T is diagnonalizable and find a basis α for V such that the matrix of T with respect to α is diagonal. **Your solution needs to be turned in at the start of class on Friday of this week, or have the receptionist in Snow 405 put your solution in my mailbox no later than 3pm on Friday.** (5 points)

Solution. Let $E \subseteq V$ be the standard basis with vectors $e_1 =: \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $e_4 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. It follows that $[T]_E^E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. A straight forward calculation shows that the characteristic polynomial $p_A(x) = (x+1)(x-1)^3$. Thus, the eigenvalues of T are -1 and 1, with algebraic multiplicities 1 and 3, respectively.

E_{-1} is the nullspace of $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Since the rank of this latter matrix is three, its nullpsace has dimension one. Thus the geometrix multiplicity of -1 is one, which equals its algebraic multiplicity. A basis for E_1 is $e_2 - e_3$, since this vector in Vcorresponds to the vector $\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$ in $\mathbb{R}^4$. In terms of matrices we have

$$T(e_2 - e_3) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -1 \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

E_1 is the nullspace of the matrix $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Since the rank of this latter matrix is one, the dimension of its nullspace is three. Therefore the geometric multiplicity of 1 equals its algebraic multiplicity. A basis for E_1 is: $e_1, e_2 + e_3, e_4$. Note that each of these matrices are symmetric so they all satisfy $T(A) = A = 1 \cdot A$.

Since the geometric multiplicity equals algebraic multiplicity for both eigenvalues, T is diagonalizable.

A basis $\alpha \subseteq V$ such that $[T]_\alpha^\alpha$ is diagonal is

$$\{e_2 - e_3, e_2, e_2 + e_3, e_4\} = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Remark. Note that E_1 is the subspace of symmetric matrices. Therefore, the space of all 2×2 symmetric matrices is a three dimensional subspace of the four dimensional space of 2×2 matrices.